NATIONAL SCIENCE TALENT CONTEST COMPANION

Volume 1: NSTC Physics Part III Multi-Step Derivations

Kinematics, rotational dynamics, relativistic cyclotrons, thermodynamic engines & LC circuits with step-by-step proofs and marking rubrics.

10	KaTeX	100%	Dual-Theme
DESCRIPTIVE DERIVATIONS	VECTOR MATH NOTATION	MARKING RUBRICS INCLUDED	MIDNIGHT & PRINT READY

PRIORITY TIER 1: EXAM CONCENTRATION — **60% WEIGHT**

> Multi-Step Proofs · > Conservation Laws · > Boundary Mechanics · > Equilibrium Systems

PRIORITY TIER 2: ANALYTICAL CORE — **40% WEIGHT**

> Differential Derivations · > Asymptotic Approximations · > First-Principles Symmetry

1. **Master Descriptive Derivations** >
2. **Drill Topical QBank** >
3. **Battle Swarm FLP Arena**

VOLUME SYLLABUS SECTIONS CHECKLIST

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P-01

Buoyancy, Equilibrium & Oscillations

30 Marks Total

Floating Cylinder with Frog Departure & Simple Harmonic Motion

OFFICIAL PROBLEM STATEMENT

A frog is riding on the top of a cylindrical piece of wood floating in still water. Half of the wood, with a diameter of $4\ \mathrm{cm}$ and length $20\ \mathrm{cm}$, is immersed in water. The density of water is $1\ \mathrm{g/cm^3}$.

a) What is the total mass of the wood along with the frog?
b) After the frog slowly leaves the wood, only one-third of the wood remains immersed. Calculate the mass of the frog.
c) Calculate x, the distance between the water level and the center of the circular end of the wooden piece when the frog is aboard, and after the frog departs.
d) Describe the motion of the wood after the instant the frog departs into the water, and derive the equation for $x(t)$ along with its oscillation frequency.

Step 1: System Geometry & Submerged Volume Calculation

The cylindrical piece of wood has radius $R = \frac{4\ \mathrm{cm}}{2} = 2\ \mathrm{cm}$ and length $L = 20\ \mathrm{cm}$. The total cross-sectional area of the cylinder is:

$$A = \pi R^2 = \pi(2)^2 = 4\pi \approx 12.566\ \mathrm{cm}^2$$

The total volume of the wooden cylinder is:

$$V_{\text{total}} = A \cdot L = 4\pi \times 20 = 80\pi \approx 251.327\ \mathrm{cm}^3$$

When half of the wood is immersed in water, the submerged volume is:

$$V_{\text{sub1}} = \frac{1}{2} V_{\text{total}} = 40\pi \approx 125.664\ \mathrm{cm}^3$$

Step 2: Total Mass of Wood and Frog (Part a)

By Archimedes' principle, the buoyant force equals the total weight of the floating system in static equilibrium:

$$F_b = (m_{\text{wood}} + m_{\text{frog}})g = \rho_w V_{\text{sub1}} g$$

Dividing through by g and substituting $\rho_w = 1.0\ \mathrm{g/cm^3}$:

$$m_{\text{total}} = m_{\text{wood}} + m_{\text{frog}} = \rho_w V_{\text{sub1}} = 1.0 \times 40\pi = 40\pi\ \mathrm{g} \approx 125.66\ \mathrm{g}$$

Thus, the combined mass of the wood and the frog is exactly $40\pi\ \mathrm{g} \approx 125.66\ \mathrm{g}$.

Step 3: Mass of Wood and Isolated Mass of Frog (Part b)

After the frog slowly departs, only one-third of the wood remains immersed. The new submerged volume is:

$$V_{\text{sub2}} = \frac{1}{3}V_{\text{total}} = \frac{80\pi}{3}\ \text{cm}^3$$

The wood floats alone in static equilibrium, so its buoyant force balances its own weight:

$$m_{\text{wood}} = \rho_w V_{\text{sub2}} = 1.0 \times \frac{80\pi}{3} = \frac{80\pi}{3}\ \text{g} \approx 83.78\ \text{g}$$

The density of the wooden piece is:

$$\rho_{\text{wood}} = \frac{m_{\text{wood}}}{V_{\text{total}}} = \frac{1}{3}\ \text{g/cm}^3 \approx 0.333\ \text{g/cm}^3$$

The mass of the frog is the difference between total initial mass and isolated wood mass:

$$m_{\text{frog}} = m_{\text{total}} - m_{\text{wood}} = 40\pi - \frac{80\pi}{3} = \frac{40\pi}{3}\ \text{g} \approx 41.89\ \text{g}$$

Step 4: Distance x from Waterline to Cylinder Axis (Part c)

Case 1 (With frog): Exactly half the cylinder is immersed. The waterline coincides with the horizontal diameter passing through the center of the circular cross-section. Therefore, the vertical distance from the water level to the center of the circular end is:

$$x_1 = 0\ \text{cm}$$

Case 2 (Without frog): One-third of the cross-sectional area is submerged. Let θ be the central angle subtended by the submerged circular arc at the cylinder axis. The submerged segment area is given by the formula:

$$A_{\text{sub}} = \frac{1}{2}R^2(\theta - \sin\theta) = \frac{1}{3}\pi R^2 \implies \theta - \sin\theta = \frac{2\pi}{3} \approx 2.0944\ \text{rad}$$

Solving $\theta - \sin\theta = 2.0944$ iteratively gives $\theta \approx 2.6053\ \text{rad}$ (149.28°).
The center of the circular end lies above the waterline by distance:

$$x_2 = R\cos\left(\frac{\theta}{2}\right) = 2.0 \times \cos(1.3027\ \text{rad}) = 2.0 \times \cos(74.64^\circ) = 2.0 \times 0.2649 = 0.53\ \text{cm}$$

Thus, the cylinder axis rests $0.53\ \text{cm}$ above the water surface.

Step 5: Dynamic Oscillation Analysis & Differential Equation (Part d)

When the frog suddenly departs, the cylinder has excess buoyancy because it is momentarily immersed to half its volume while only requiring one-third. Let y denote the vertical displacement from the new equilibrium position ($y = 0$).
Near equilibrium, the waterline width is $w = 2\sqrt{R^2 - x_2^2} = 2\sqrt{2^2 - 0.53^2} = 2\sqrt{3.719} \approx 3.857 \text{ cm}$.
For small vertical displacement y, the restoring buoyant force is:

$$F_{\text{net}} = -\rho_w g(wL)y$$

Applying Newton's second law for the wood of mass m_{wood}:

$$m_{\text{wood}}\frac{d^2y}{dt^2} + (\rho_w g w L)y = 0 \implies \frac{d^2y}{dt^2} + \omega^2 y = 0$$

where the angular frequency of vertical oscillation is:

$$\omega = \sqrt{\frac{\rho_w g w L}{m_{\text{wood}}}} = \sqrt{\frac{1.0 \times 980 \times 3.857 \times 20}{83.78}} = \sqrt{\frac{75597.2}{83.78}} = \sqrt{902.33} \approx 30.04 \text{ rad/s}$$

The natural period of oscillation is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{30.04} \approx 0.209 \text{ s}$$

With initial conditions $y(0) = -0.53 \text{ cm}$ and $\dot{y}(0) = 0$, the displacement as a function of time is:

$$x(t) = 0.53 - 0.53\cos(30.04t) \text{ cm}$$

EXAMINER MARKING RUBRIC

Volume and total mass derivation with Archimedes equilibrium (Part a)	**6 Marks**
Isolated wood mass, density calculation, and frog mass calculation (Part b)	**6 Marks**
Geometric circular segment area transcendental equation and x derivation (Part c)	**8 Marks**
Equation of motion, restoring stiffness $\rho_w g w L$, and frequency ω (Part d)	**10 Marks**

OLYMPIAD INSIGHT

In buoyancy oscillations, the restoring force is proportional to the waterline cross-section wL. For a cylinder, w depends non-linearly on immersion depth, but for small perturbations, linearization around the static equilibrium position yields clean simple harmonic motion.

COPYABLE LATEX SNIPPET

```
m_{\text{total}} = \rho_w \left(\frac{1}{2}\pi R^2 L\right) = 40\pi\text{ g} \approx 125.66\text{ g}\\
m_{\text{frog}} = \frac{40\pi}{3}\text{ g} \approx 41.89\text{ g}\\
\theta - \sin\theta = \frac{2\pi}{3} \implies x_2 = R\cos(\theta/2) \approx 0.53\text{ cm}\\
\omega = \sqrt{\frac{\rho_w g w L}{m_{\text{wood}}}} \approx 30.04\text{ rad/s}, \quad T \approx 0.21\text{ s}
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P-02

Special Relativity & Electrodynamics

25 Marks Total

Relativistic Cyclotron Resonance & Lorentz Trajectory Curvature

OFFICIAL PROBLEM STATEMENT

A particle of rest mass m_0 and charge q enters a region of uniform magnetic field $\vec{B} = B\hat{z}$ with velocity $\vec{v}$ perpendicular to the field. At relativistic speeds, the particle's momentum is given by $\vec{p} = \gamma m_0 \vec{v}$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

a) Prove that the kinetic energy and speed $v = |\vec{v}|$ remain strictly constant throughout the motion.
b) Derive the relativistic orbit radius R in terms of q, B, m_0, and relativistic kinetic energy K.
c) Derive the relativistic cyclotron frequency ω_c, and explain why classical cyclotrons fail for high-energy protons.

Step 1: Energy Invariance under Magnetic Force (Part a)

The relativistic equation of motion under the Lorentz force with zero electric field is:

$$\frac{d\vec{p}}{dt} = q(\vec{v} \times \vec{B})$$

The rate of work done by the magnetic force on the particle is:

$$\frac{dE}{dt} = \vec{v} \cdot \frac{d\vec{p}}{dt} = q\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$$

Because $\vec{v} \times \vec{B}$ is identically perpendicular to $\vec{v}$, the scalar product vanishes. Since total energy $E = \gamma m_0 c^2$ is constant, γ and the speed $v = |\vec{v}|$ are strictly conserved constants of motion.

Step 2: Relativistic Orbital Radius Derivation (Part b)

Since speed v is constant, $\frac{d\gamma}{dt} = 0$, and the momentum derivative simplifies to:

$$\frac{d\vec{p}}{dt} = \gamma m_0 \frac{d\vec{v}}{dt} = \gamma m_0 \vec{a}$$

In uniform circular motion of radius R, the centripetal acceleration is $a_c = \frac{v^2}{R}$. Equating magnitudes:

$$\gamma m_0 \frac{v^2}{R} = qvB \implies R = \frac{\gamma m_0 v}{qB} = \frac{p}{qB}$$

From relativistic energy-momentum relation $E^2 = p^2c^2 + m_0^2c^4$, and total energy $E = K + m_0c^2$:

$$p^2c^2 = (K + m_0c^2)^2 - m_0^2c^4 = K^2 + 2Km_0c^2 = K(K + 2m_0c^2)$$

Taking the square root and substituting into the radius formula:

$$R = \frac{\sqrt{K(K + 2m_0c^2)}}{qBc}$$

In the non-relativistic limit $K \ll m_0c^2$, this reduces to $R \approx \frac{\sqrt{2m_0K}}{qB} = \frac{m_0v}{qB}$, recovering the classical result.

Step 3: Relativistic Cyclotron Frequency & Synchrocyclotron Moat (Part c)

The orbital angular velocity is $\omega_c = \frac{v}{R}$. Substituting $R = \frac{\gamma m_0 v}{qB}$:

$$\omega_c = \frac{qB}{\gamma m_0} = \frac{qB}{m_0}\sqrt{1 - \frac{v^2}{c^2}} = \frac{qBc^2}{E}$$

In a classical cyclotron, the radio-frequency alternating electric field has constant frequency $\omega_{\mathrm{RF}} = \frac{qB}{m_0}$. As protons gain kinetic energy, γ increases above 1.0, which causes ω_c to decrease. The accelerating particles fall out of phase with the fixed RF gaps and undergo deceleration rather than acceleration. This fundamental relativistic desynchronization necessitates synchrocyclotrons (modulating ω_{RF} downward over time) or isochronous cyclotrons (shaping radial magnetic field $B(r) = \gamma(r)B_0$).

EXAMINER MARKING RUBRIC

Proof of zero work by magnetic force and conservation of γ and speed (Part a)	**6 Marks**
Relativistic equation of motion, centripetal balance, and radius in terms of K (Part b)	**10 Marks**
Derivation of $\omega_c = qB/(\gamma m_0)$ and physical breakdown analysis of classical cyclotrons (Part c)	**9 Marks**

OLYMPIAD INSIGHT

Magnetic fields do no work on charged particles, preserving $|\vec{v}|$ and γ. However, orbital inertia increases by factor γ, widening orbit radius and reducing cyclotron frequency.

COPYABLE LATEX SNIPPET

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\frac{dE}{dt} = q \vec{v} \cdot (\vec{v} \times \vec{B}) = 0 \implies v = \text{const}\\
R = \frac{p}{qB} = \frac{\sqrt{K(K + 2m_0 c^2)}}{qBc}\\
\omega_c = \frac{v}{R} = \frac{qB}{\gamma m_0} = \frac{qBc^2}{E}
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P-03

Rigid Body Rotational Dynamics

20 Marks Total

Rolling Without Slipping on an Inclined Plane: Shape Factor Comparison

OFFICIAL PROBLEM STATEMENT

A uniform rigid body of mass M, radius R, and moment of inertia $I_{\text{cm}} = \beta M R^2$ rolls down a rough inclined plane of angle θ without slipping.

a) Derive the linear acceleration a_{cm} of the center of mass in terms of g, θ, and the dimensionless shape factor β.

b) Derive the minimum coefficient of static friction μ_s required to maintain pure rolling without slipping.

c) Compare the acceleration and minimum friction requirements for a solid sphere ($\beta = 2/5$), a solid cylinder ($\beta = 1/2$), and a thin cylindrical hoop ($\beta = 1$).

Step 1: Free Body Diagram & Equations of Motion

Choose an axis parallel to the incline pointing downward. The forces acting on the body are:

1. Gravity: component down the incline $Mg\sin\theta$, and normal to the incline $Mg\cos\theta$.
2. Normal force: $N = Mg\cos\theta$ perpendicular to the incline.
3. Static friction: force f_s acting upward along the incline at the contact point.

Translational equation of motion:

$$Mg\sin\theta - f_s = Ma_{\text{cm}}$$

Rotational equation of motion about the center of mass:

$$\tau_{\text{cm}} = f_s R = I_{\text{cm}}\alpha = (\beta M R^2)\alpha$$

The no-slip kinematic condition enforces $a_{\text{cm}} = R\alpha$, so:

$$f_s R = \beta M R^2 \left(\frac{a_{\text{cm}}}{R}\right) \implies f_s = \beta M a_{\text{cm}}$$

Step 2: Center-of-Mass Acceleration Derivation (Part a)

Substitute the friction expression $f_s = \beta M a_{\text{cm}}$ into the translational equation:

$$Mg\sin\theta - \beta M a_{\text{cm}} = Ma_{\text{cm}}$$

Factoring out mass M and grouping terms:

$$g\sin\theta = (1+\beta)a_{\text{cm}} \implies a_{\text{cm}} = \frac{g\sin\theta}{1+\beta}$$

Notice that acceleration is completely independent of mass M and radius R; it depends solely on the geometric mass distribution parameter β.

Step 3: Minimum Static Friction Coefficient (Part b)

The required static friction force to sustain pure rolling is:

$$f_s = \beta M a_{\text{cm}} = \beta M \left(\frac{g \sin\theta}{1+\beta} \right) = \frac{\beta}{1+\beta} M g \sin\theta$$

To prevent slipping, the required static friction cannot exceed the maximum available static friction:

$$f_s \le f_{s,\max} = \mu_s N = \mu_s M g \cos\theta$$

Therefore:

$$\frac{\beta}{1+\beta} M g \sin\theta \le \mu_s M g \cos\theta \implies \mu_s \ge \left(\frac{\beta}{1+\beta} \right) \tan\theta$$

Step 4: Comparative Hierarchy for Classical Geometries (Part c)

Evaluating each geometry with its respective shape factor β:
1. Solid Sphere ($\beta = 2/5 = 0.40$):

$$a_{\text{sphere}} = \frac{g \sin\theta}{1 + 2/5} = \frac{5}{7} g \sin\theta \approx 0.714 g \sin\theta, \quad \mu_{s,\min} = \frac{2}{7} \tan\theta \approx 0.286 \tan\theta$$

2. Solid Cylinder ($\beta = 1/2 = 0.50$):

$$a_{\text{cylinder}} = \frac{g \sin\theta}{1 + 1/2} = \frac{2}{3} g \sin\theta \approx 0.667 g \sin\theta, \quad \mu_{s,\min} = \frac{1}{3} \tan\theta \approx 0.333 \tan\theta$$

3. Thin Cylindrical Hoop ($\beta = 1.00$):

$$a_{\text{hoop}} = \frac{g \sin\theta}{1 + 1} = \frac{1}{2} g \sin\theta = 0.500 g \sin\theta, \quad \mu_{s,\min} = \frac{1}{2} \tan\theta = 0.500 \tan\theta$$

Ranking from fastest to slowest: Solid Sphere $>$ Solid Cylinder $>$ Cylindrical Hoop. The body with mass concentrated nearest the rotational axis has lowest β, dedicating less potential energy to rotational inertia and more to translational speed.

EXAMINER MARKING RUBRIC

Translational and rotational equations with no-slip kinematic condition (Part a)	**6 Marks**
Rigorous derivation of $a_{\text{cm}} = g \sin\theta/(1+\beta)$ (Part a)	**4 Marks**
Derivation of static friction condition $\mu_s \ge \frac{\beta}{1+\beta} \tan\theta$ (Part b)	**5 Marks**
Numerical evaluation and physical ranking of sphere, cylinder, and hoop (Part c)	**5 Marks**

OLYMPIAD INSIGHT

Rolling motion partitions gravitational potential energy between translation $\frac{1}{2}Mv^2$ and rotation $\frac{1}{2}I\omega^2 = \frac{1}{2}\beta M v^2$. A lower shape factor β directs a larger fraction of energy into linear velocity.

COPYABLE LATEX SNIPPET

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M g \sin\theta - f_s = M a_{\text{cm}}, \quad f_s R = \beta M R^2 (a_{\text{cm}}/R)\\
a_{\text{cm}} = \frac{g \sin\theta}{1 + \beta}\\
f_s = \frac{\beta}{1 + \beta} M g \sin\theta \le \mu_s M g \cos\theta \implies \mu_s \ge \frac{\beta}{1 + \beta} \tan\th
eta
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P-04

Thermodynamics & Statistical Mechanics

25 Marks Total

Carnot Cycle Efficiency & Finite-Time Thermodynamic Entropy

OFFICIAL PROBLEM STATEMENT

An ideal gas undergoes a reversible Carnot cycle consisting of two isothermal processes (at hot reservoir T_H and cold reservoir T_C) and two reversible adiabatic processes.

a) Using the ideal gas equation of state $PV = nRT$ and adiabatic relation $PV^\gamma = \text{const}$, derive the thermal efficiency $\eta = 1 - \frac{T_C}{T_H}$.
b) Prove that the cyclic integral of $\frac{\delta Q}{T}$ vanishes identically: $\oint \frac{\delta Q_{\text{rev}}}{T} = 0$, establishing entropy S as a state function.
c) For an irreversible engine with finite heat transfer rate $\dot{Q}_H = \alpha(T_H - T_{\text{in}})$ where T_{in} is working fluid temperature, explain why real efficiencies fall below the Carnot ceiling.

Step 1: Isothermal and Adiabatic Path Relations (Part a)

The four strokes of the cycle are:
1. Isothermal expansion ($A \to B$) at temperature T_H from volume V_A to V_B:

$$Q_H = W_{AB} = \int_{V_A}^{V_B} \frac{nRT_H}{V} dV = nRT_H \ln\left(\frac{V_B}{V_A}\right)$$

2. Reversible adiabatic expansion ($B \to C$) from T_H to T_C:

$$T_H V_B^{\gamma-1} = T_C V_C^{\gamma-1} \implies \frac{T_H}{T_C} = \left(\frac{V_C}{V_B}\right)^{\gamma-1}$$

3. Isothermal compression ($C \to D$) at temperature T_C from V_C to V_D:

$$Q_C = -W_{CD} = -\int_{V_C}^{V_D} \frac{nRT_C}{V} dV = nRT_C \ln\left(\frac{V_C}{V_D}\right)$$

4. Reversible adiabatic compression ($D \to A$) from T_C to T_H:

$$T_C V_D^{\gamma-1} = T_H V_A^{\gamma-1} \implies \frac{T_H}{T_C} = \left(\frac{V_D}{V_A}\right)^{\gamma-1}$$

Step 2: Volume Ratio Equivalence & Carnot Efficiency Derivation

Equating the temperature-volume ratios from processes 2 and 4:

$$\left(\frac{V_C}{V_B}\right)^{\gamma-1} = \left(\frac{V_D}{V_A}\right)^{\gamma-1} \implies \frac{V_C}{V_B} = \frac{V_D}{V_A} \implies \frac{V_B}{V_A} = \frac{V_C}{V_D}$$

Now compute the thermal efficiency $\eta = \frac{W_{\text{net}}}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}$:

$$\frac{Q_C}{Q_H} = \frac{nRT_C \ln(V_C/V_D)}{nRT_H \ln(V_B/V_A)} = \frac{T_C}{T_H}$$

Therefore, the theoretical Carnot efficiency is:

$$\eta_{\text{Carnot}} = 1 - \frac{T_C}{T_H}$$

Step 3: Clausius Theorem & Entropy Invariance (Part b)

Summing the heat over temperature ratios across the entire cycle:

$$\oint \frac{\delta Q_{\text{rev}}}{T} = \frac{Q_H}{T_H} + \frac{-Q_C}{T_C} = \frac{nRT_H \ln(V_B/V_A)}{T_H} - \frac{nRT_C \ln(V_C/V_D)}{T_C} = nR\ln\left(\frac{V_B}{V_A}\right) - nR\ln\left(\frac{V_C}{V_D}\right) = 0$$

Because the cyclic integral vanishes for any arbitrary reversible path, the differential $dS = \frac{\delta Q_{\text{rev}}}{T}$ is an exact differential, proving that entropy S is a thermodynamic state variable independent of path.

Step 4: Finite-Time Thermodynamics & Curzon-Ahlborn Limit (Part c)

A reversible Carnot cycle produces zero net power because infinite time is required for heat to transfer across zero temperature difference ($T_H - T_{\text{in}} = 0$). For finite heat flow $\dot{Q}_H = \alpha(T_H - T_{\text{in}})$, a temperature difference is required, meaning $T_{\text{in}} < T_H$ and $T_{\text{out}} > T_C$. The maximum power output occurs at the Curzon-Ahlborn efficiency:

$$\eta_{\text{CA}} = 1 - \sqrt{\frac{T_C}{T_H}} < \eta_{\text{Carnot}}$$

This demonstrates that real heat engines are bounded by non-equilibrium entropy generation at finite operating rates.

EXAMINER MARKING RUBRIC

Formulation of 4 cycle stages and volume ratio equality $V_B/V_A = V_C/V_D$ (Part a)	**8 Marks**
Derivation of thermal efficiency $\eta = 1 - T_C/T_H$ (Part a)	**5 Marks**
Clausius cyclic integral evaluation proving entropy exactness (Part b)	**6 Marks**
Finite-rate thermal resistance analysis and Curzon-Ahlborn derivation (Part c)	**6 Marks**

OLYMPIAD INSIGHT

Carnot efficiency represents an unattainable upper bound at zero power output. Any realistic thermal engine operating at non-zero throughput requires finite temperature differentials across thermal interfaces, generating irreversible entropy.

COPYABLE LATEX SNIPPET

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Q_H = nRT_H \ln(V_B/V_A), \quad Q_C = nRT_C \ln(V_C/V_D)\\
\frac{T_H}{T_C} = (V_C/V_B)^{\gamma-1} = (V_D/V_A)^{\gamma-1} \implies V_B/V_A = V_C/V_D\\
\eta = 1 - \frac{Q_C}{Q_H} = 1 - \frac{T_C}{T_H}, \quad \oint \frac{\delta Q}{T} = 0
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P-05

Electromagnetism & Oscillations

20 Marks Total

Electromagnetic LC Circuit Oscillations & Energy Dynamics

OFFICIAL PROBLEM STATEMENT

An ideal circuit consists of an inductor with inductance L and a capacitor with capacitance C. At time $t = 0$, the capacitor holds an initial charge Q_0 and the current is zero.

a) Using Kirchhoff's loop rule and energy conservation, derive the second-order differential equation governing charge $Q(t)$.
b) Find the explicit solutions for charge $Q(t)$, current $I(t)$, and oscillation frequency ω_0.
c) Show that the total electromagnetic energy $U_{\text{total}} = U_E(t) + U_B(t)$ is conserved at all times, and compute the maximum current $I_{\max}$.

Step 1: Kirchhoff's Loop Rule & Differential Equation (Part a)

Traversing the loop clockwise, moving across the discharging capacitor from the negative to positive plate provides a potential rise $V_C = \frac{Q}{C}$. Moving across the inductor in the direction of current I gives a potential drop $-L\frac{dI}{dt}$. By Kirchhoff's voltage law:

$$\frac{Q}{C} - L\frac{dI}{dt} = 0$$

Since current represents the rate of discharge of the capacitor plate, $I = -\frac{dQ}{dt}$, which gives $\frac{dI}{dt} = -\frac{d^2Q}{dt^2}$. Substituting this in:

$$\frac{Q}{C} - L\left(-\frac{d^2Q}{dt^2}\right) = 0 \implies L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0 \implies \frac{d^2Q}{dt^2} + \omega_0^2 Q = 0$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ is the natural resonance frequency.

Alternatively, by conservation of energy, the total electromagnetic energy stored in the ideal circuit is constant:

$$E = \frac{Q^2}{2C} + \frac{1}{2}LI^2 \implies \frac{dE}{dt} = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0$$

Substituting $\frac{dQ}{dt} = -I$ and $\frac{dI}{dt} = -\frac{d^2Q}{dt^2}$:

$$-I\left(\frac{Q}{C} + L\frac{d^2Q}{dt^2}\right) = 0 \implies L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0$$

Step 2: Explicit Solutions for Charge and Current (Part b)

The general solution to the harmonic oscillator differential equation is:

$$Q(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

Applying initial conditions at $t = 0$:
1. $Q(0) = Q_0 \implies A = Q_0$
2. $I(0) = -\left.\frac{dQ}{dt}\right|_{t=0} = -\omega_0 B = 0 \implies B = 0$

Thus, the charge oscillations are:

$$Q(t) = Q_0\cos(\omega_0 t)$$

The current in the circuit is:

$$I(t) = -\frac{dQ}{dt} = Q_0\omega_0\sin(\omega_0 t) = I_{\max}\sin(\omega_0 t)$$

where $I_{\max} = Q_0\omega_0 = \frac{Q_0}{\sqrt{LC}}$.

Step 3: Electromagnetic Energy Conservation (Part c)

The total energy stored in the circuit is the sum of the electrostatic energy in the capacitor and magnetic energy in the inductor:

$$U_E(t) = \frac{Q^2(t)}{2C} = \frac{Q_0^2\cos^2(\omega_0 t)}{2C}$$

$$U_B(t) = \frac{1}{2}LI^2(t) = \frac{1}{2}L\left(Q_0^2\omega_0^2\sin^2(\omega_0 t)\right) = \frac{1}{2}L\left(\frac{Q_0^2}{LC}\right)\sin^2(\omega_0 t) = \frac{Q_0^2\sin^2(\omega_0 t)}{2C}$$

Adding the two energies together:

$$U_{\text{total}} = U_E(t) + U_B(t) = \frac{Q_0^2}{2C}\left[\cos^2(\omega_0 t) + \sin^2(\omega_0 t)\right] = \frac{Q_0^2}{2C} = \text{constant}$$

The energy continuously oscillates between purely electrostatic when $Q = \pm Q_0$ ($I = 0$) and purely magnetic when $I = \pm I_{\max}$ ($Q = 0$).

EXAMINER MARKING RUBRIC

Formulation of Kirchhoff's voltage law and derivation of differential equation (Part a)	**6 Marks**
Solving with boundary conditions for $Q(t)$, $I(t)$, and natural frequency ω_0 (Part b)	**7 Marks**
Energy conservation proof and derivation of peak current $I_{\max} = Q_0/\sqrt{LC}$ (Part c)	**7 Marks**

OLYMPIAD INSIGHT

The LC circuit is the electromagnetic dual of the mass-spring mechanical oscillator: inductance L mirrors mass m (inertia against current changes), inverse capacitance $1/C$ mirrors spring constant k, and charge Q mirrors displacement x.

COPYABLE LATEX SNIPPET

```
L \frac{d^2 Q}{dt^2} + \frac{1}{C} Q = 0 \implies \omega_0 = \frac{1}{\sqrt{LC}}\\
Q(t) = Q_0 \cos(\omega_0 t), \quad I(t) = \frac{Q_0}{\sqrt{LC}} \sin(\omega_0 t)\\
U_{\text{total}} = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{Q_0^2}{2C} = \text{const}
```

P-06

Gravitation & Potential Theory

25 Marks Total

Gravitational Potential & Escape Velocity for Non-Uniform Planet

OFFICIAL PROBLEM STATEMENT

Consider a spherical planet of radius R and total mass M whose density decreases linearly from center to surface according to:

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right), \quad 0 \leq r \leq R$$

a) Determine the central density ρ_0 in terms of total mass M and radius R.
b) Derive the gravitational field strength $g(r)$ for all points inside the planet ($r \leq R$) and find the radius $r_{\max}$ where $g(r)$ reaches its maximum.
c) Derive the gravitational potential $\Phi(0)$ at the center of the planet and determine the escape velocity v_{esc} for a particle launched from the surface.

Step 1: Total Mass Integration & Central Density ρ_0 (Part a)

Integrate density over spherical shells of volume $dV = 4\pi r^2 dr$:

$$M = \int_0^R \rho(r) 4\pi r^2 dr = 4\pi\rho_0 \int_0^R \left(r^2 - \frac{r^3}{R}\right) dr$$

Computing the definite integral:

$$M = 4\pi\rho_0 \left[\frac{R^3}{3} - \frac{R^4}{4R}\right] = 4\pi\rho_0 R^3 \left(\frac{1}{3} - \frac{1}{4}\right) = 4\pi\rho_0 R^3 \left(\frac{1}{12}\right) = \frac{\pi\rho_0 R^3}{3}$$

Solving for central density ρ_0:

$$\rho_0 = \frac{3M}{\pi R^3} = 4\bar{\rho}$$

where $\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$ is the average density of the planet. The central density is exactly 4 times the mean planetary density.

Step 2: Interior Gravitational Field & Maximum Field Location (Part b)

By Gauss's law for gravity, only mass enclosed within radius r contributes to the gravitational field:

$$M(r) = 4\pi\rho_0 \int_0^r \left(r'^2 - \frac{r'^3}{R}\right) dr' = 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) = \frac{4\pi\rho_0 r^3}{3}\left(1 - \frac{3r}{4R}\right)$$

Substituting $\rho_0 = \frac{3M}{\pi R^3}$:

$$M(r) = 4M\left(\frac{r}{R}\right)^3 \left(1 - \frac{3r}{4R}\right)$$

The gravitational field strength at interior distance r is:

$$g(r) = \frac{GM(r)}{r^2} = \frac{4GMr}{R^3}\left(1 - \frac{3r}{4R}\right) = \frac{4GM}{R^3}\left(r - \frac{3r^2}{4R}\right)$$

To find the radius of maximum gravitational field, take the derivative with respect to r and set to zero:

$$\frac{dg}{dr} = \frac{4GM}{R^3}\left(1 - \frac{6r}{4R}\right) = 0 \implies 1 - \frac{3r}{2R} = 0 \implies r_{\max} = \frac{2}{3}R$$

The maximum gravitational field inside the planet is:

$$g_{\max} = g\left(\frac{2}{3}R\right) = \frac{4GM}{R^3}\left(\frac{2}{3}R - \frac{3(4/9R^2)}{4R}\right) = \frac{4GM}{R^2}\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{4}{3}\frac{GM}{R^2} = \frac{4}{3}g_{\text{surface}}$$

Step 3: Central Potential & Surface Escape Velocity (Part c)

Gravitational potential is defined with $\Phi(\infty) = 0$. At the surface $r = R$, the potential is standard Keplerian:

$$\Phi(R) = -\frac{GM}{R}$$

The surface escape velocity requires total mechanical energy $E = \frac{1}{2}mv_{\text{esc}}^2 + m\Phi(R) = 0$:

$$v_{\text{esc}} = \sqrt{-2\Phi(R)} = \sqrt{\frac{2GM}{R}}$$

Notice that surface escape velocity is completely identical to a uniform planet of equal mass M because the exterior potential depends only on total enclosed mass.
To find the central potential $\Phi(0)$, integrate the radial field $\frac{d\Phi}{dr} = g(r)$ from the center to the surface:

$$\Phi(R) - \Phi(0) = \int_0^R g(r)dr = \frac{4GM}{R^3}\int_0^R \left(r - \frac{3r^2}{4R}\right) dr$$

Evaluating the interior integral:

$$\int_0^R \left(r - \frac{3r^2}{4R}\right) dr = \left[\frac{R^2}{2} - \frac{R^3}{4R}\right] = \frac{R^2}{4}$$

Therefore:

$$\Phi(R) - \Phi(0) = \frac{4GM}{R^3}\left(\frac{R^2}{4}\right) = \frac{GM}{R} \implies \Phi(0) = \Phi(R) - \frac{GM}{R} = -\frac{GM}{R} - \frac{GM}{R} = -\frac{2GM}{R}$$

The potential at the center is exactly twice the surface potential.

EXAMINER MARKING RUBRIC

Volume integration of linear density profile and central density $\rho_0 = 3M/(\pi R^3)$ (Part a)	**6 Marks**
Gauss's law interior mass $M(r)$, field $g(r)$, and maximum at $r = 2R/3$ (Part b)	**10 Marks**
Surface escape velocity $v_{\text{esc}} = \sqrt{2GM/R}$ and potential integration $\Phi(0) = -2GM/R$ (Part c)	**9 Marks**

OLYMPIAD INSIGHT

Because exterior field is spherically symmetric, Gauss's law guarantees that surface escape velocity depends solely on total mass M and radius R, completely insensitive to the radial internal distribution.

COPYABLE LATEX SNIPPET

```
M = \int_0^R 4\pi r^2 \rho_0 (1 - r/R) dr = \frac{\pi \rho_0 R^3}{3} \implies \rho_0 = \frac{3M}{\pi R^3}\\
g(r) = \frac{4GM}{R^3}(r - 3r^2/(4R)) \implies r_{\text{max}} = \frac{2}{3}R, \quad g_{\text{max}} = \frac{4}{3}\frac{G
M}{R^2}\\
\Phi(0) = -\frac{2GM}{R}, \quad v_{\text{esc}} = \sqrt{\frac{2GM}{R}}
```

P-07

Wave Optics & Interference

20 Marks Total

Dielectric Thin-Film Phase Shift in Young's Double-Slit Interference

OFFICIAL PROBLEM STATEMENT

In a Young's double-slit experiment, two narrow slits separated by distance d are illuminated by monochromatic light of wavelength λ. The interference pattern is viewed on a screen placed at distance D ($D \gg d$). A transparent dielectric sheet of refractive index n and uniform thickness t is placed over one of the slits.

a) Derive the optical path difference introduced between the two rays by the presence of the film.
b) Derive the linear displacement Δy of the central bright fringe on the screen.
c) If the central fringe shifts to the position originally occupied by the m-th bright fringe, derive the thickness t in terms of m, λ, and n.

Step 1: Optical Path Difference Introduced by Dielectric Film (Part a)

When light travels through vacuum or air (refractive index ≈ 1) across thickness t, the optical path length is $L_1 = t$. When it travels through the dielectric medium of refractive index n across thickness t, the speed of light is reduced to $v = c/n$. The optical path length (defined as the equivalent vacuum distance for the same phase accumulation) is:

$$L_2 = nt$$

Therefore, inserting the plate increases the optical path of that ray by:

$$\Delta L_{\text{film}} = L_2 - L_1 = (n-1)t$$

The corresponding phase shift introduced to the wave is:

$$\Delta\phi = \frac{2\pi}{\lambda}(n-1)t$$

Step 2: Fringe Shift Derivation on Distant Screen (Part b)

In the absence of the plate, the optical path difference at height y on the screen is $\Delta r = d\sin\theta \approx d\frac{y}{D}$. With the dielectric film placed over the upper slit (slit 1), the total optical path difference at point P is:

$$\Delta r_{\text{total}} = (r_2 - r_1) - (n-1)t = d\frac{y}{D} - (n-1)t$$

The central bright fringe occurs where the net optical path difference is strictly zero (zero phase difference):

$$\Delta r_{\text{total}} = 0 \implies d\frac{y_0}{D} - (n-1)t = 0$$

Solving for the position y_0 of the new central fringe:

$$y_0 = \frac{D}{d}(n-1)t$$

The fringe pattern shifts upward toward the side containing the dielectric sheet by displacement:

$$\Delta y = \frac{D}{d}(n-1)t$$

Step 3: Film Thickness in Terms of Fringe Count (Part c)

The fringe width (spacing between adjacent bright fringes) is:

$$\beta = \frac{\lambda D}{d}$$

If the central maximum shifts to the original position of the m-th bright fringe, the total displacement is:

$$\Delta y = m\beta = m\frac{\lambda D}{d}$$

Equating the two expressions for Δy:

$$\frac{D}{d}(n-1)t = m\frac{\lambda D}{d} \implies (n-1)t = m\lambda$$

Solving for the thickness of the dielectric film:

$$t = \frac{m\lambda}{n-1}$$

This principle forms the operational basis of Jamin and Rayleigh interferometers used to measure gas refractive indices to six decimal places.

EXAMINER MARKING RUBRIC

Optical path length concept and derivation of $\Delta L = (n-1)t$ (Part a)	**6 Marks**
Formulation of net path difference and derivation of fringe shift $\Delta y = \frac{D}{d}(n-1)t$ (Part b)	**8 Marks**
Relation to fringe count m and derivation of thickness formula $t = m\lambda/(n-1)$ (Part c)	**6 Marks**

OLYMPIAD INSIGHT

Inserting a transparent medium adds an optical path delay $(n-1)t$. To restore zero optical path difference, the ray from the uncovered slit must travel further, tilting the interference centroid toward the covered slit.

COPYABLE LATEX SNIPPET

```
\Delta L_{\text{film}} = (n - 1)t, \quad \Delta \phi = \frac{2\pi}{\lambda}(n - 1)t\\
\Delta r_{\text{net}} = d\frac{y}{D} - (n - 1)t = 0 \implies \Delta y = \frac{D}{d}(n - 1)t\\
t = \frac{m \lambda}{n - 1}
```

P-08

Fluid Dynamics & Viscosity

20 Marks Total

Stokes' Viscous Drag & Terminal Velocity in Creeping Flow

OFFICIAL PROBLEM STATEMENT

A small spherical bead of mass m, radius r, and density ρ_p falls vertically under gravity through a viscous fluid of density ρ_f and dynamic viscosity η. At low Reynolds numbers ($Re \ll 1$), the fluid resistance is governed by Stokes' law $F_{\text{drag}} = 6\pi\eta r v$.

a) Write the complete differential equation of motion for the falling sphere and solve for velocity $v(t)$ assuming release from rest at $t = 0$.
b) Derive the asymptotic terminal velocity v_t and express it in terms of r, g, η, ρ_p, and ρ_f.
c) Define the characteristic relaxation time τ of the velocity growth and compute the time required to achieve 99% of terminal velocity.

Step 1: Net Force Formulation & Equation of Motion (Part a)

Taking downward as positive, the three forces acting on the falling bead are:
1. Gravitational force: $F_g = mg = \frac{4}{3}\pi r^3 \rho_p g$
2. Buoyant force: $F_b = \frac{4}{3}\pi r^3 \rho_f g$
3. Viscous drag: $F_d = 6\pi\eta r v$

Newton's second law gives:

$$m\frac{dv}{dt} = F_g - F_b - F_d = \frac{4}{3}\pi r^3(\rho_p - \rho_f)g - 6\pi\eta r v$$

Dividing by mass $m = \frac{4}{3}\pi r^3 \rho_p$:

$$\frac{dv}{dt} = g\left(1 - \frac{\rho_f}{\rho_p}\right) - \frac{6\pi\eta r}{\frac{4}{3}\pi r^3 \rho_p} v = g\left(\frac{\rho_p - \rho_f}{\rho_p}\right) - \frac{9\eta}{2r^2\rho_p} v$$

Define characteristic time constant $\tau = \frac{2r^2\rho_p}{9\eta}$ and effective gravitational acceleration $g_{\text{eff}} = g\left(\frac{\rho_p - \rho_f}{\rho_p}\right)$:

$$\frac{dv}{dt} + \frac{v}{\tau} = g_{\text{eff}}$$

Step 2: Integration for Time-Dependent Velocity $v(t)$

Separating variables with initial condition $v(0) = 0$:

$$\int_0^v \frac{dv'}{g_{\text{eff}} - \frac{v'}{\tau}} = \int_0^t dt \implies -\tau \ln\left(1 - \frac{v}{g_{\text{eff}}\tau}\right) = t$$

Exponentiating both sides:

$$1 - \frac{v}{g_{\text{eff}}\tau} = e^{-t/\tau} \implies v(t) = g_{\text{eff}}\tau\left(1 - e^{-t/\tau}\right)$$

As $t \to \infty$, $e^{-t/\tau} \to 0$, giving the terminal velocity $v_t = g_{\text{eff}}\tau$.

Step 3: Terminal Velocity & Relaxation Time Evaluation (Parts b and c)

Multiplying g_{eff} and τ:

$$v_t = g_{\text{eff}}\tau = \left[g\left(\frac{\rho_p - \rho_f}{\rho_p}\right)\right]\left[\frac{2r^2\rho_p}{9\eta}\right] = \frac{2r^2 g(\rho_p - \rho_f)}{9\eta}$$

This is the classical Stokes' settling velocity equation. Notice that $v_t \propto r^2$, meaning larger particles settle far faster than smaller aerosols.
The time required to reach 99% of terminal velocity satisfies:

$$\frac{v(t)}{v_t} = 1 - e^{-t_{99}/\tau} = 0.99 \implies e^{-t_{99}/\tau} = 0.01 \implies t_{99} = -\tau\ln(0.01) = \tau\ln(100) \approx 4.605\tau$$

Thus, terminal velocity is effectively reached in less than five relaxation time constants.

EXAMINER MARKING RUBRIC

Formulation of effective buoyancy and differential equation (Part a)	**6 Marks**
Exact integration for velocity transient $v(t) = v_t(1 - e^{-t/\tau})$ (Part a)	**6 Marks**
Derivation of terminal velocity $v_t = \frac{2r^2 g(\rho_p - \rho_f)}{9\eta}$ (Part b)	**5 Marks**
Evaluation of time to 99% terminal velocity ($t_{99} \approx 4.61\tau$) (Part c)	**3 Marks**

OLYMPIAD INSIGHT

Stokes' terminal velocity scales as the square of the radius (r^2), while mass scales as r^3. This quadratic scaling makes microscopic droplets suspend in air almost indefinitely while gravel falls rapidly.

COPYABLE LATEX SNIPPET

```
m \frac{dv}{dt} = \frac{4}{3}\pi r^3 (\rho_p - \rho_f)g - 6\pi \eta r v\\
v(t) = v_t \left(1 - e^{-t/\tau}\right), \quad \tau = \frac{2r^2\rho_p}{9\eta}\\
v_t = \frac{2 r^2 g (\rho_p - \rho_f)}{9 \eta}, \quad t_{99} = \tau \ln(100) \approx 4.61\tau
```

P-09

Celestial Mechanics & Gravitation

25 Marks Total

Gravitational Tidal Gradient & The Fluid Roche Limit

OFFICIAL PROBLEM STATEMENT

A small fluid satellite of mass m, radius r, and mean density ρ_m orbits a massive primary planet of mass M, radius R, and density ρ_M at orbital distance d ($d \gg r$).

a) Using Taylor series expansion, derive the differential gravitational tidal acceleration Δg_{tidal} exerted across the diameter of the satellite.
b) In the non-rotating frame, derive the classical rigid Roche limit d_R where tidal force overcomes self-gravitation at the satellite's sub-planetary surface point.
c) Explain why liquid or loosely bound rubble-pile satellites have a higher disintegration limit ($d \approx 2.44R(\rho_M/\rho_m)^{1/3}$) compared to rigid monolithic rocks.

Step 1: Differential Tidal Acceleration Derivation (Part a)

The primary planet of mass M is centered at origin. The satellite's center of mass is at distance d. The gravitational acceleration at the satellite's center of mass is $g_0 = \frac{GM}{d^2}$.
At the near point of the satellite's surface facing the planet ($r_{\text{near}} = d - r$):

$$g(d-r) = \frac{GM}{(d-r)^2} = \frac{GM}{d^2\left(1-\frac{r}{d}\right)^2} = \frac{GM}{d^2}\left(1-\frac{r}{d}\right)^{-2}$$

Using binomial expansion for $r \ll d$:

$$\left(1-\frac{r}{d}\right)^{-2} = 1 + 2\frac{r}{d} + 3\left(\frac{r}{d}\right)^2 + \dots$$

The tidal acceleration relative to the satellite's center of mass pulling outward toward the planet is:

$$a_{\text{tidal}} = g(d-r) - g_0 \approx \frac{GM}{d^2}\left(1+2\frac{r}{d}\right) - \frac{GM}{d^2} = \frac{2GMr}{d^3}$$

Equivalently, this is the first derivative of the gravitational field: $a_{\text{tidal}} = -\left|\frac{dg}{dr}\right| r = \frac{2GM}{d^3} r$.

Step 2: Rigid Roche Limit Derivation (Part b)

Consider a test particle of mass μ resting on the surface of the satellite at the sub-planetary point. The outward tidal force pulling it toward the primary planet is:

$$F_{\text{tidal}} = \mu a_{\text{tidal}} = \frac{2GM\mu r}{d^3}$$

The inward self-gravitational force holding the test particle to the satellite is:

$$F_{\text{self}} = \frac{Gm\mu}{r^2}$$

Tidal disruption begins when the outward tidal force exceeds the inward self-gravitational attraction:

$$F_{\text{tidal}} \geq F_{\text{self}} \implies \frac{2GM\mu r}{d^3} \geq \frac{Gm\mu}{r^2} \implies \frac{2Mr}{d^3} \geq \frac{m}{r^2} \implies d^3 \leq 2\frac{M}{m}r^3$$

Taking the cube root gives the rigid Roche limit:

$$d_R = r\left(2\frac{M}{m}\right)^{1/3}$$

Now substitute planetary masses in terms of radii and densities: $M = \frac{4}{3}\pi R^3 \rho_M$ and $m = \frac{4}{3}\pi r^3 \rho_m$:

$$\frac{M}{m} = \left(\frac{R}{r}\right)^3 \left(\frac{\rho_M}{\rho_m}\right)$$

Substituting back into the Roche formula:

$$d_R = r\left[2\left(\frac{R}{r}\right)^3 \left(\frac{\rho_M}{\rho_m}\right)\right]^{1/3} = 2^{1/3} R \left(\frac{\rho_M}{\rho_m}\right)^{1/3} \approx 1.260R\left(\frac{\rho_M}{\rho_m}\right)^{1/3}$$

Step 3: Fluid vs Rigid Roche Limit & Planetary Rings (Part c)

In a liquid or rubble-pile satellite, the body elongates into a prolate ellipsoid under tidal stress. The elongation moves the tidal tips closer to the planet and farther from the satellite's center of mass, weakening self-gravity while strengthening tidal pull. Detailed hydrodynamic integration by Édouard Roche (including centrifugal rotation) shifts the numerical prefactor from 1.26 to:

$$d_{\text{fluid}} \approx 2.44R\left(\frac{\rho_M}{\rho_m}\right)^{1/3}$$

Because $d_{\text{fluid}} \approx 2.44R$ lies outside the primary planet's surface for similar densities, any fluid body orbiting within this radius is torn apart into planetary rings, exactly as observed in Saturn's rings.

EXAMINER MARKING RUBRIC

Taylor expansion and derivation of tidal gradient $a_{\text{tidal}} = 2GMr/d^3$ (Part a)	**8 Marks**
Force balance between self-gravity and tidal pull to find $d_R = R(2\rho_M/\rho_m)^{1/3}$ (Part b)	**10 Marks**
Physical explanation of fluid prolate deformation and derivation of 2.44 factor (Part c)	**7 Marks**

OLYMPIAD INSIGHT

Tidal forces scale as inverse cube d^{-3} because they are differential forces across a finite radius. Inside the Roche limit, gravitational binding energy is negative relative to the tidal field, converting moons into planetary rings.

COPYABLE LATEX SNIPPET

```
a_{\text{tidal}} = \frac{d g}{d r} r = \frac{2 G M r}{d^3}\\
\frac{2 G M \mu r}{d^3} = \frac{G m \mu}{r^2} \implies d_R = R \left(2 \frac{\rho_M}{\rho_m}\right)^{1/3} \approx 1.26 R
\left(\frac{\rho_M}{\rho_m}\right)^{1/3}\\
d_{\text{fluid}} \approx 2.44 R \left(\frac{\rho_M}{\rho_m}\right)^{1/3}
```

P-10

Alternating Current & Circuit Resonance

20 Marks Total

Series RLC Resonance, Complex Impedance & Bandwidth Derivation

OFFICIAL PROBLEM STATEMENT

A series RLC circuit is driven by an alternating sinusoidal voltage source $\mathcal{E}(t) = V_0 \cos(\omega t)$. The circuit contains a resistor R, an inductor L, and a capacitor C.

a) Using complex phasors, derive the complex impedance $Z(\omega)$ and the current amplitude $I_0(\omega)$.
b) Derive the resonance frequency ω_0 where current amplitude reaches its peak, and calculate the phase angle ϕ at resonance.
c) Derive the half-power frequencies ω_1 and ω_2 (where average power drops to half its resonance value), and prove that bandwidth is $\Delta\omega = \frac{R}{L}$.

Step 1: Complex Impedance & Steady-State Current (Part a)

In the complex domain, the applied voltage is $\tilde{V}(t) = V_0 e^{j\omega t}$. The impedance of each component is:

$$Z_R = R, \quad Z_L = j\omega L, \quad Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

Since the elements are in series, the total complex impedance is:

$$Z(\omega) = Z_R + Z_L + Z_C = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The magnitude of the impedance is:

$$|Z(\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The phase angle ϕ between voltage and current is:

$$\tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

The steady-state current amplitude is:

$$I_0(\omega) = \frac{V_0}{|Z(\omega)|} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Step 2: Resonance Condition & Zero Phase (Part b)

Current amplitude is maximized when impedance $|Z(\omega)|$ is minimized. Since R^2 is constant, minimization occurs when reactance is identically zero:

$$X(\omega) = \omega L - \frac{1}{\omega C} = 0 \implies \omega_0^2 L = \frac{1}{C} \implies \omega_0 = \frac{1}{\sqrt{LC}}$$

At resonance:

$$Z(\omega_0) = R, \quad I_0(\omega_0) = \frac{V_0}{R}, \quad \tan\phi = 0 \implies \phi = 0$$

The current and voltage are strictly in phase at resonance, and the circuit behaves as a pure ohmic resistor.

Step 3: Half-Power Frequencies & Bandwidth $\Delta\omega = R/L$ (Part c)

Average power dissipated in the resistor is $P(\omega) = \frac{1}{2} I_0^2(\omega) R$. At resonance, $P_{\max} = \frac{V_0^2}{2R}$. At the half-power points $\omega_{1,2}$, $P(\omega) = \frac{1}{2} P_{\max}$, which requires:

$$I_0(\omega) = \frac{I_0(\omega_0)}{\sqrt{2}} \implies |Z(\omega)| = \sqrt{2}R$$

Substituting $|Z(\omega)|^2 = 2R^2$:

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2 \implies \omega L - \frac{1}{\omega C} = \pm R$$

Dividing through by L:

$$\omega^2 \mp \frac{R}{L}\omega - \omega_0^2 = 0$$

Taking the physical positive root for each sign:

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_0^2}, \quad \omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_0^2}$$

Subtracting ω_1 from ω_2 gives the $-3\ \mathrm{dB}$ bandwidth:

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L}$$

The quality factor Q of the resonator is:

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

EXAMINER MARKING RUBRIC

Formulation of complex phasors and impedance magnitude $\lvert Z(\omega)\rvert$ (Part a)	**6 Marks**
Derivation of resonance frequency $\omega_0 = 1/\sqrt{LC}$ and zero phase condition (Part b)	**6 Marks**
Half-power condition $\lvert Z\rvert = \sqrt{2}R$ and quadratic solution for bandwidth $\Delta\omega = R/L$ (Part c)	**8 Marks**

OLYMPIAD INSIGHT

Bandwidth $\Delta\omega = R/L$ depends exclusively on resistance and inductance, independent of capacitance. The capacitor sets the central tuning frequency, while the resistor governs the dissipation rate and peak sharpness.

COPYABLE LATEX SNIPPET

```
Z(\omega) = R + j\left(\omega L - \frac{1}{\omega C}\right) \implies I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/(\omega C))^2}}\\
\omega_0 = \frac{1}{\sqrt{LC}}, \quad Z(\omega_0) = R\\
\omega L - \frac{1}{\omega C} = \pm R \implies \Delta\omega = \omega_2 - \omega_1 = \frac{R}{L}, \quad Q = \frac{\omega_0 L}{R}
```

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ESSENTIAL NUMERICAL VALUES & CONSTANTS

SPEED OF LIGHT (C) **2.9979 x 10^8 m/s**	PLANCK CONSTANT (H) **6.626 x 10^-34 J s**	ELEMENTARY CHARGE (E) **1.602 x 10^-19 C**	PERMITTIVITY (EPS_0) **8.854 x 10^-12 F/m**
PERMEABILITY (MU_0) **4pi x 10^-7 H/m**	PROTON MASS (M_P) **1.6726 x 10^-27 kg**	GRAVITATIONAL CONST (G) **6.674 x 10^-11 N m^2/kg^2**	UNIVERSAL GAS CONST (R) **8.314 J/(mol K)**

1. **Derivations Mastered**
2. **QBank Accuracy > 85%**
3. **FLP Screening Qualified**